

# Fourth-order four-point boundary value problem on time scales

Ilkay Yaslan Karaca

*Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey*

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## Abstract

In this work, we consider a fourth-order four-point boundary value problem for dynamic equations on time scales. We establish criteria for the existence of a solution and a positive solution by using the Leray–Schauder fixed point theorem. We also give an example to illustrate our results.

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## 1. Introduction

We are concerned with the following fourth-order four-point boundary value problem on time scales  $\mathbb{T}$ :

$$\begin{cases} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = f(t, y(\sigma(t)), y^{\Delta^2}(t)), \\ y(\sigma^4(b)) = 0, \quad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \quad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0, \end{cases} \quad (1.1)$$

for  $t \in [a, b] \subset \mathbb{T}$ ,  $a \leq \xi_1 < \xi_2 \leq \sigma(b)$ , and  $f \in \mathcal{C}([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ .

We will assume that the following conditions are satisfied.

(H1)  $\alpha, \beta, \gamma, \delta, \zeta, \eta \geq 0$ .

(H2)  $q(t) \geq 0$ . If  $q(t) \equiv 0$ , then  $\gamma + \zeta > 0$ .

(H3)  $d = \beta + \alpha(\sigma^4(b) - a) > 0$ ,  $p = \gamma\eta + \delta\zeta + \gamma\zeta(\xi_2 - \xi_1) > 0$ ,  $\delta - \gamma\xi_1 \geq 0$ ,  $\eta - \zeta\sigma^3(b) + \zeta\xi_2 \geq 0$ .

Throughout this work we let  $\mathbb{T}$  be any time scale (nonempty closed subset of  $\mathbb{R}$ ), and  $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ . Some preliminary definitions and theorems on time scales can be found in the books [4,5] which are excellent references for the calculus of time scales.

It is well known that boundary value problems for second-order dynamic equations have received considerable attention [1,3,7,8,12]. However, there are fewer results in the literature on boundary value problems for fourth-order ordinary differential equations when  $\mathbb{T} = \mathbb{R}$  [2,6,9,10,13].

The purpose of this work is to establish some existence criteria of solution and positive solution for the BVP (1.1) by using the Leray–Schauder fixed point theorem. We cite some appropriate references here: [6,9]. Liu [9] has studied

*E-mail address:* [ilkay.karaca@ege.edu.tr](mailto:ilkay.karaca@ege.edu.tr).

the BVP (1.1) for when  $\mathbb{T} = \mathbb{R}$ ,  $a = \beta = \delta = \eta = \xi_1 = 0$ ,  $b = \alpha = \gamma = \zeta = \xi_2 = 1$ ,  $q(t) \equiv 0$ . He has obtained sufficient conditions for the existence of positive solutions by using the Krasnoselskii fixed point theorem. Chen, Ni and Wang [6] have established a sufficient condition for the existence of a positive solution of the BVP (1.1) for when  $\mathbb{T} = \mathbb{R}$ ,  $a = \beta = 0$ ,  $b = \alpha = 1$ ,  $q(t) \equiv 0$  and  $f(t, y(\sigma(t)), y^{\Delta^2}(\sigma(t))) = f(t, y(t))$  by using the upper and lower solutions method. Our problem is more general than the fourth-order four-point boundary value problem in [6] and the method used in this work is as in [11]. So our results are even new for differential equations, as well as for dynamic equations on general time scales.

## 2. Preliminaries

Denote by  $\varphi$  and  $\psi$  the solutions of the corresponding homogeneous equation:

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = 0, \quad t \in [\xi_1, \rho(\xi_2)] \quad (2.1)$$

under the initial conditions

$$\begin{cases} \varphi(\xi_1) = \eta, & \varphi^{\Delta}(\xi_1) = \zeta, \\ \psi(\xi_2) = \delta, & \psi^{\Delta}(\xi_2) = -\gamma. \end{cases} \quad (2.2)$$

Define the number  $D$  by

$$D := \zeta \psi(\xi_1) - \eta \psi^{\Delta}(\xi_1) = \delta \varphi^{\Delta}(\xi_2) + \gamma \varphi(\xi_2). \quad (2.3)$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for  $\varphi$  and  $\psi$  the following equations:

$$\varphi(t) = \eta + \zeta(t - \xi_1) + \int_{\xi_1}^t \int_{\xi_1}^{\tau} q(s)\varphi(\sigma(s))\Delta s \Delta \tau, \quad (2.4)$$

$$\psi(t) = \delta + \gamma(\xi_2 - t) + \int_t^{\xi_2} \int_{\tau}^{\xi_2} q(s)\psi(\sigma(s))\Delta s \Delta \tau. \quad (2.5)$$

**Lemma 2.1.** *Under the conditions (H1)–(H2) the following inequalities:*

$$\begin{cases} \varphi(t) \geq 0, t \in [\xi_1, \sigma(\xi_2)]; & \psi(t) \geq 0, t \in [\xi_1, \xi_2]; \\ \varphi^{\Delta}(t) \geq 0, t \in [\xi_1, \xi_2]; & \psi^{\Delta}(t) \leq 0, t \in [\xi_1, \xi_2] \end{cases} \quad (2.6)$$

are yielded.

**Proof.** Using the induction method on time scales as in [3] one can easily see these inequalities (2.6).  $\square$

**Lemma 2.2.** *Under the conditions (H1)–(H2) the inequality  $D > 0$  holds.*

**Proof.** By (2.3) and (2.4) we have

$$D = \gamma\eta + \delta\zeta + \gamma\zeta(\xi_2 - \xi_1) + \eta \int_{\xi_1}^{\xi_2} q(s)\varphi(\sigma(s))\Delta s + \zeta \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\tau} q(s)\varphi(\sigma(s))\Delta s \Delta \tau. \quad (2.7)$$

Since  $\varphi(t) \geq 0$  for  $t \in [\xi_1, \sigma(\xi_2)]$ , from (2.7) we have

$$D \geq \gamma\eta + \delta\zeta + \gamma\zeta(\xi_2 - \xi_1). \quad (2.8)$$

If  $q(t) \equiv 0$ , then in (2.8) the equality holds. From the condition (H1), we get  $D > 0$ . This proof is completed.  $\square$

**Lemma 2.3.** *Assume that the conditions (H1)–(H2) are satisfied. If  $h \in \mathcal{C}[\xi_1, \rho(\xi_2)]$ , then the boundary value problem*

$$\begin{cases} y^{\Delta^2}(t) - q(t)y(\sigma(t)) = h(t), & t \in [\xi_1, \rho(\xi_2)], \\ \gamma y(\xi_1) - \delta y^{\Delta}(\xi_1) = 0, \\ \zeta y(\xi_2) + \eta y^{\Delta}(\xi_2) = 0 \end{cases} \quad (2.9)$$

has a unique solution

$$y(t) = - \int_{\xi_1}^{\xi_2} G(t, s) h(s) \Delta s, \quad (2.10)$$

where

$$G(t, s) = \frac{1}{D} \begin{cases} \psi(\sigma(s))\varphi(t), & t \leq s, \\ \psi(t)\varphi(\sigma(s)), & t \geq \sigma(s). \end{cases} \quad (2.11)$$

Here  $D$ ,  $\varphi$ ,  $\psi$  are as in (2.3)–(2.5) respectively.

**Proof.** It is easy to see that the general solution of the equation

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = h(t), \quad t \in [\xi_1, \rho(\xi_2)]$$

has the form

$$y(t) = c_1\varphi(t) + c_2\psi(t) - \frac{1}{D} \int_{\xi_1}^t [\varphi(\sigma(s))\psi(t) - \varphi(t)\psi(\sigma(s))]h(s) \Delta s,$$

where  $c_1$  and  $c_2$  are arbitrary constants. Substituting this expression for  $y(t)$  in the boundary conditions of BVP (2.9) we can evaluate  $c_1$  and  $c_2$ . After some easy calculations we can get (2.10) and (2.11).  $\square$

**Lemma 2.4.** Under the conditions (H1)–(H2) the Green's function of BVP (2.9) possesses the following property:

$$G(t, s) \geq 0, \quad (t, s) \in [\xi_1, \xi_2] \times [\xi_1, \rho(\xi_2)].$$

**Proof.** The lemma follows from (2.11), Lemmas 2.1 and 2.2 immediately.  $\square$

**Lemma 2.5.** Assume that the conditions (H1)–(H2) are satisfied. If  $h \in \mathcal{C}[a, b]$ , then the boundary value problem

$$\begin{cases} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = h(t), & t \in [a, b], \\ y(\sigma^4(b)) = 0, & \alpha y(a) - \beta y^{\Delta}(a) = 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, & \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0 \end{cases} \quad (2.12)$$

has a unique solution

$$y(t) = \int_a^{\sigma^4(b)} G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) h(s) \Delta s \Delta \xi,$$

where

$$G_1(t, s) = \frac{1}{d} \begin{cases} (\sigma^4(b) - \sigma(s))(\alpha(t - a) + \beta), & t \leq s, \\ (\sigma^4(b) - t)(\alpha(\sigma(s) - a) + \beta), & t \geq \sigma(s), \end{cases} \quad (2.13)$$

and

$$G_2(t, s) = \frac{1}{D} \begin{cases} \psi(\sigma(s))\varphi(t), & t \leq s, \\ \psi(t)\varphi(\sigma(s)), & t \geq \sigma(s). \end{cases} \quad (2.14)$$

Here  $D$ ,  $\varphi$ ,  $\psi$  are as in (2.3)–(2.5) respectively,  $d = \beta + \alpha(\sigma^4(b) - a)$ .

**Proof.** Let us consider the following BVP:

$$\begin{cases} y^{\Delta^2}(t) = - \int_{\xi_1}^{\xi_2} G_2(t, s) h(s) \Delta s, & t \in [a, \sigma^2(b)], \\ y(\sigma^4(b)) = 0, & \alpha y(a) - \beta y^{\Delta}(a) = 0. \end{cases} \quad (2.15)$$

The Green's function associated with the BVP (2.15) is  $G_1(t, s)$ . This completes the proof.  $\square$

**Lemma 2.6.** Assume that the conditions (H1)–(H3) are satisfied. If  $y$  satisfies

$$\begin{aligned} y^{\Delta^4} - q(t)y^{\Delta^2}(\sigma(t)) &\geq 0, \quad t \in [a, b], \\ y(\sigma^4(b)) &\geq 0, \quad \alpha y(a) - \beta y^{\Delta}(a) \geq 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) &\leq 0, \quad \text{and} \quad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) \leq 0, \end{aligned}$$

then  $y(t) \geq 0, t \in [a, \sigma^4(b)]$  and  $y^{\Delta^2}(t) \leq 0, t \in [a, \sigma^2(b)]$ .

**Proof.** Let

$$\begin{cases} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = h(t), & t \in [a, b], \\ y(\sigma^4(b)) = t_0, & \alpha y(a) - \beta y^{\Delta}(a) = t_1, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = t_2, & \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = t_3, \end{cases}$$

where  $t_0 \geq 0, t_1 \geq 0, t_2 \leq 0, t_3 \leq 0, h \geq 0$ .

It is easy to check that  $y$  and  $y^{\Delta^2}$  can be given by the expression

$$\begin{aligned} y(t) &= S(t) - \int_a^{\sigma^4(b)} G_1(t, \xi) R(\xi) \Delta \xi + \int_a^{\sigma^4(b)} G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) h(s) \Delta s \Delta \xi, \\ y^{\Delta^2}(t) &= R(t) - \int_{\xi_1}^{\xi_2} G_2(t, s) h(s) \Delta s, \end{aligned}$$

where

$$\begin{aligned} S(t) &= \frac{1}{d} \{ [\alpha(t - a) + \beta] t_0 + (\sigma^4(b) - t) t_1 \}, \\ R(t) &= \frac{1}{p} \{ [\zeta(\xi_2 - t) + \eta] t_2 + [\gamma(t - \xi_1) + \delta] t_3 \}, \end{aligned}$$

and  $G_1(t, s), G_2(t, s)$  are as in (2.13) and (2.14) respectively. The hypothesis of the lemma implies that  $S(t) \geq 0$  for  $t \in [a, \sigma^4(b)]$ ,  $R(t) \leq 0$  for  $t \in [a, \sigma^3(b)]$ ,  $G_1(t, s) \geq 0$  for  $(t, s) \in [a, \sigma^4(b)] \times [a, \sigma^3(b)]$ ,  $G_2(t, s) \geq 0$  for  $(t, s) \in [a, \sigma^3(b)] \times [a, \sigma^2(b)]$ . Therefore we get  $y(t) \geq 0$  for  $t \in [a, \sigma^4(b)]$  and  $y^{\Delta^2}(t) \leq 0$  for  $t \in [a, \sigma^2(b)]$ . The proof is completed.  $\square$

### 3. Main results

In this section, we establish some existence criteria for a solution and a positive solution by using the Leray–Schauder fixed point theorem.

**Theorem 3.1.** Assume that the conditions (H1)–(H2) are satisfied,  $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exist  $c > 0$  and  $0 < k \leq 1/m[1 - 1/(2(b - a))]$  such that

$$\max\{|f(t, y, z)| : t \in [a, b], |y| \leq c(b - a), |z| \leq kc(b - a)\} \leq kc(b - a)/n,$$

where  $b - a > 1/2$ ,  $m = \max_{t \in [a, \sigma^4(b)]} \int_a^{\sigma^2(b)} |G_1(t, \xi)| \Delta \xi$ , and  $n = \max_{t \in [a, \sigma^2(b)]} \int_{\xi_1}^{\xi_2} |G_2(t, \xi)| \Delta \xi$ . Then, the BVP (1.1) has at least one solution  $y^*$  satisfying

$$|y^*(t)| \leq c(b - a), \quad t \in [a, \sigma^4(b)], \quad \text{and} \quad |y^{*\Delta^2}(t)| \leq kc(b - a), \quad t \in [a, \sigma^2(b)].$$

**Proof.** Let

$$A = \{y|y : [a, \sigma^4(b)] \rightarrow \mathbb{R} \text{ is continuous}\} \quad \text{and} \quad B = \{z|z : [a, \sigma^2(b)] \rightarrow \mathbb{R} \text{ is continuous}\}$$

be equipped with their norms  $|y|_1 = \max_{t \in [a, \sigma^4(b)]} |y(t)|$  and  $|z|_2 = \max_{t \in [a, \sigma^2(b)]} |z(t)|$ , respectively. If  $C = A \times B$  and, for any  $(y, z) \in C$ , its norm

$$\|(y, z)\| = \max \left\{ |y|_1, \frac{1}{k} |z|_2 \right\},$$

then  $C$  is a Banach space.

If  $y^{\Delta^2}(t) = z(t)$  for  $t \in [a, \sigma^2(b)]$ , then the BVP (1.1) is equivalent to the following system of integral equations:

$$\begin{cases} y(t) = \int_a^{\sigma^4(b)} G_1(t, s)z(s)\Delta s, & t \in [a, \sigma^4(b)], \\ z(t) = -\int_{\xi_1}^{\xi_2} G_2(t, s)f(s, y(s), z(s))\Delta s, & t \in [a, \sigma^2(b)]. \end{cases} \quad (3.1)$$

Define an operator  $\psi : C \rightarrow C$ :

$$\psi(y, z) = (\psi_1(y, z), \psi_2(y, z)),$$

where  $\psi_1(y, z)(t) = \int_a^{\sigma^4(b)} G_1(t, s)z(s)\Delta s$ ,  $t \in [a, \sigma^4(b)]$  and  $\psi_2(y, z)(t) = \int_{\xi_1}^{\xi_2} G_2(t, s)f(s, y(s), z(s))\Delta s$ ,  $t \in [a, \sigma^2(b)]$ . Then system (3.1), and so also the BVP (1.1), is equivalent to the fixed point equation

$$\psi(y, z) = (y, z), \quad (y, z) \in C.$$

Moreover, it is easy to see that  $\psi : C \rightarrow C$  is completely continuous.

If  $C_h = \{(y, z) \in C : \|(y, z)\| \leq h\}$  where  $h = c(b-a)$ , then  $C_h$  is a closed convex subset of  $C$ . Suppose that  $(y, z) \in C_h$ ; then  $|y|_1 \leq h$  and  $|z|_2 \leq kh$ . So

$$|y(t)| \leq h, \quad t \in [a, \sigma^4(b)] \quad (3.2)$$

and

$$|z(t)| \leq kh, \quad t \in [a, \sigma^2(b)], \quad (3.3)$$

which imply that

$$|f(t, y, z)| \leq kc(b-a)/n, \quad t \in [a, b]. \quad (3.4)$$

From (3.3) and  $0 < k \leq 1/m[1 - 1/(2(b-a))]$ , we have

$$\begin{aligned} |\psi(y, z)|_1 &= \max_{t \in [a, \sigma^4(b)]} \left| \int_a^{\sigma^4(b)} G_1(t, s)z(s)\Delta s \right| \leq \max_{t \in [a, \sigma^4(b)]} \int_a^{\sigma^4(b)} |G_1(t, s)z(s)|\Delta s \\ &\leq kc(b-a) \max_{t \in [a, \sigma^4(b)]} \int_a^{\sigma^4(b)} |G_1(t, s)|\Delta s = kcm(b-a) \leq c(b-a). \end{aligned} \quad (3.5)$$

On the other hand, it follows from (3.4) that

$$\begin{aligned} |\psi(y, z)|_2 &= \max_{t \in [a, \sigma^2(b)]} \left| \int_{\xi_1}^{\xi_2} G_2(t, s)f(s, y(s), z(s))\Delta s \right| \leq \max_{t \in [a, \sigma^2(b)]} \int_{\xi_1}^{\xi_2} |G_2(t, s)f(s, y(s), z(s))|\Delta s \\ &\leq kc(b-a)/n \max_{t \in [a, \sigma^2(b)]} \int_{\xi_1}^{\xi_2} |G_2(t, s)|\Delta s = kc(b-a). \end{aligned} \quad (3.6)$$

In view of (3.5) and (3.6), we know that

$$\|\psi(y, z)\| = \max \left\{ |\psi_1(y, z)|_1, \frac{1}{k} |\psi_2(y, z)|_2 \right\} \leq c(b-a) = h,$$

which shows that  $\psi : C_h \rightarrow C_h$ . Then it follows from the Leray–Schauder fixed point theorem that  $\psi$  has a fixed point  $(y^*, z^*) \in C_h$ . In other words the BVP (1.1) has one solution  $y^*$  satisfying

$$|y^*(t)| \leq c(b-a), \quad t \in [a, \sigma^4(b)] \quad \text{and} \quad |y^{*\Delta^2}(t)| \leq kc(b-a), \quad t \in [a, \sigma^2(b)]. \quad \square$$

**Theorem 3.2.** Assume that the conditions (H1)–(H3) are satisfied,  $f : [a, b] \times \mathbb{R}^+ \times \mathbb{R}^- \rightarrow \mathbb{R}$  is continuous, and there exist  $c > 0$  and  $0 < k \leq 1/m[1 - 1/(2(b-a))]$  such that

$$\max\{f(t, y, z) : t \in [a, b], 0 \leq y \leq c(b-a), -kc(b-a) \leq z \leq 0\} \leq kc(b-a)/n,$$

where  $b - a > 1/2$ ,  $m = \max_{t \in [a, \sigma^4(b)]} \int_a^{\sigma^2(b)} |G_1(t, \xi)| \Delta \xi$ ,  $n = \max_{t \in [a, \sigma^2(b)]} \int_{\xi_1}^{\xi_2} |G_2(t, \xi)| \Delta \xi$ ,  $\mathbb{R}^+ = [0, \infty)$ , and  $\mathbb{R}^- = (-\infty, 0]$ . Then, the BVP (1.1) has at least one solution

$$0 \leq y^*(t) \leq c(b - a), \quad t \in [a, \sigma^4(b)] \quad \text{and} \quad -kc(b - a) \leq z^*(t) \leq 0, \quad t \in [a, \sigma^2(b)].$$

**Proof.** Let

$$f_1(t, y, z) = \begin{cases} f(t, y, z), & (t, y, z) \in [a, b] \times \mathbb{R}^+ \times \mathbb{R}^-, \\ f(t, y, 0), & (t, y, z) \in [a, b] \times \mathbb{R}^+ \times \mathbb{R}^+, \end{cases}$$

and

$$f_2(t, y, z) = \begin{cases} f_1(t, y, z), & (t, y, z) \in [a, b] \times \mathbb{R}^+ \times \mathbb{R}, \\ f_1(t, 0, z), & (t, y, z) \in [a, b] \times \mathbb{R}^- \times \mathbb{R}. \end{cases}$$

Then  $f_2 : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  is continuous and

$$\begin{aligned} & \max\{|f_2(t, y, z)| : t \in [a, b], |y| \leq c(b - a), |z| \leq kc(b - a)\} \\ &= \max\{f(t, y, z) : t \in [a, b], 0 \leq y \leq c(b - a), -kc(b - a) \leq z \leq 0\} \leq kc(b - a)/n. \end{aligned}$$

Consider the BVP

$$\begin{cases} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = f_2(t, y(\sigma(t)), y^{\Delta^2}(t)), \\ y(\sigma^4(b)) = 0, \quad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \quad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0. \end{cases} \quad (3.7)$$

By Theorem 3.1, we know that the BVP (3.7) has one solution  $y^*$  satisfying

$$|y^*(t)| \leq c(b - a), \quad t \in [a, \sigma^4(b)], \quad \text{and} \quad |y^{*\Delta^2}(t)| \leq kc(b - a), \quad t \in [a, \sigma^2(b)].$$

By Lemma 2.6,  $y^*(t) \geq 0$ ,  $t \in [a, \sigma^4(b)]$  and  $y^{*\Delta^2}(t) \leq 0$ ,  $t \in [a, \sigma^2(b)]$ . So  $f_2(t, y^*(t), y^{*\Delta^2}(t)) = f(t, y^*(t), y^{*\Delta^2}(t))$ , for  $t \in [a, b]$ . Therefore,  $y^*$  is a solution of the BVP (1.1) and satisfies

$$0 \leq y^*(t) \leq c(b - a), \quad t \in [a, \sigma^4(b)] \quad \text{and} \quad -kc(b - a) \leq y^{*\Delta^2}(t) \leq 0, \quad t \in [a, \sigma^2(b)]. \quad \square$$

**Corollary 3.1.** Assume that all the conditions of Theorem 3.2 are fulfilled. If  $f(t, 0, 0)$  is not identically zero for  $t \in [a, b]$ , then the BVP (1.1) has one positive solution.

**Proof.** By Theorem 3.2,  $y^*$  is a solution of the BVP (1.1) satisfying  $0 \leq y^*(t) \leq c(b - a)$ ,  $t \in [a, \sigma^4(b)]$  and  $-kc(b - a) \leq y^{*\Delta^2}(t) \leq 0$ ,  $t \in [a, \sigma^2(b)]$ . Since  $G_2(t, s) > 0$  for  $(t, s) \in [a, \sigma^3(b)] \times [a, \sigma^2(b)]$  and  $f(t, 0, 0)$  is not identically zero for  $t \in [a, b]$ , we get  $y^{*\Delta^2}(t) = -\int_{\xi_1}^{\xi_2} G_2(t, s)f(s, y^*(\sigma(s)), y^{*\Delta^2}(\sigma(s)))\Delta s < 0$ . Therefore for  $t \in (a, \sigma^4(b))$  we have  $y^*(t) = \int_a^{\sigma^4(b)} G_1(t, s)y^{*\Delta^2}(s)\Delta s > 0$  which shows that  $y^*$  is a positive solution of the BVP (1.1).  $\square$

**Example 3.1.** We illustrate Corollary 3.1 with the specific time scale  $\mathbb{T} = [-2, 2] \cup [3, 6]$ . Consider the BVP

$$\begin{cases} y^{\Delta^4}(t) - y^{\Delta^2}(\sigma(t)) = f(t, y(\sigma(t)), y^{\Delta^2}(\sigma(t))), & t \in [-1, 5], \\ y(5) = 0, \quad y(-1) - 2y^{\Delta}(-1) = 0, \\ y^{\Delta^3}(0) = 0, \quad y^{\Delta^2}(4) + y^{\Delta^3}(4) = 0, \end{cases} \quad (3.8)$$

where  $f(t, y, z) = \frac{1}{2+k^2}(y^2 + z^2 + 0.09)$ ,  $(t, y, z) \in [-1, 5] \times \mathbb{R}^+ \times \mathbb{R}^-$ . It is clear that  $m = 9.696428571$ ,  $n = 2.057890738$ . Thus, if we choose  $c = 0.05$  and  $k = \frac{3}{10}n$ , then all the conditions of Corollary 3.1 are satisfied. It follows from Corollary 3.1 that the BVP (3.8) has at least one positive solution.

## References

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